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Generalized order statistics from generalized exponential distributions in explicit forms

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In the present paper, we study the Generalized Order Statistics (GOS) from Generalized Exponential Distribution (GED) and Linear Exponential Distributions (LED) in explicit forms. We obtain, the joint distribution, distribution of single, and conditional distribution of two GOS from GED and LED. Several interesting special cases, when GOS are from Exponential distribution have been discussed.

keywords: Generalized Exponential Distribution, Linear Exponential Distribution, Generalized Order Statistics, Conditional Distribution.

1 Introduction

In probability theory and statistics, the exponential distribution is a family of continuous probability distributions. It describes the time between events in a Poisson. In other words, it is a process in which events occur continuously and independently at a constant average rate. The ordered random variables such as order statistics play an important roles in many branches of statistics and applied probability. The concept of Generalized Order Statistics (GOS) is introduced in Kamps (1995) and showed that order statistics, record values, and some other ordered random variables can be considered as special cases of generalized order statistics. Several recurrence relations satisfied by the single and the product moments for order statistics from the Generalized Exponential Distribution (GED) are discussed in Ragab (2004). The relationships can be written in terms of polygamma and hypergeometric functions and used in a simple recursive manner in

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order to compute the single and the product moments of all order statistics for all sample sizes. The GED with two non-negative parameters θ and λ is considered to be one of those distributions which have real attention from researchers. GED has a right skewed unimodal density function and monotone hazard function similar to the density functions and hazard functions of the gamma and Weibull distributions. It is observed that it can be used quite effectively to analyze lifetime data in place of gamma, Weibull and log-normal distributions, for example see Gupta and Kundu (1999). The Linear Exponential Distribution (LED) has been used in the area of reliability and life-testing see, for example see Bain (1974).

Definition 1.1 The random variables $X(1, n, \widetilde{m}, k), \ldots, X(n, n, \widetilde{m}, k)$ are called GOS based on the cumulative distribution function (*cdf*), if their joint probability density function (*pdf*) is given by Kamps (1995).

$$f(x_1, \ldots, x_n) = k\left(\prod_{j=1}^{n-1} \gamma_j\right)\left[\prod_{i=1}^{n-1}(1 - F(x_i))^{m_i} f(x_i)\right](1 - F(x_n))^{k-1} f(x_n), \quad (1)$$

on the cone $F^{-1}(0) < X_1 \le X_2 \le \ldots \le X_n < F^{-1}(1)$ of $\mathbb{R}^n$, with parameters $n \in \mathbb{N},\ n \ge 2,\ k > 0,\ \widetilde{m} = (m_1,\ m_2, \ldots m_{n-1}) \in \mathbb{R}^{n-1},\ M_r = \sum_{j=r}^{n-1} m_j$, such that $\gamma_r = k + n - r + M_r > 0$ for all $r \in \{1, 2, \ldots, n-1\}$, let $c_{r-1} = \prod_{j=1}^{r} \gamma_j,\ r = 1, 2, \ldots, n-1$ and $\gamma_n = k$.

Definition 1.2 The random variable X is said to have GED with two non-negative parameters θ and λ, if it's *pdf* and *cdf* are given, respectively, by Mahmoud and Al-Nagar (2009).

$$f(x; \theta, \lambda) = \theta\lambda \exp(-\lambda x)\left[1 - \exp(-\lambda x)\right]^{\theta-1},\ x > 0 \quad (2)$$

$$F(x; \theta, \lambda) = \left[1 - \exp(-\lambda x)\right]^{\theta},\ x > 0 \quad (3)$$

Note that the Exponential distribution with parameter λ is a special case of the GED when $\theta = 1$.

Definition 1.3 A random variable X is said to have a LED with two non-negative parameters θ and λ, if it's *pdf* and *cdf* are given, respectively, by Mahmoud and Al-Nagar (2009).

$$f(x; \theta, \lambda) = (\lambda + \theta x)\exp\left(-\left[\lambda x + \frac{\theta x^2}{2}\right]\right),\ x > 0 \quad (4)$$

$$F(x;\theta,\lambda) = 1-\exp\left(-\left[\lambda x+\frac{\theta x^2}{2}\right]\right),\ x>0 \tag{5}$$

This paper is organized as follows: Sections 2 and 3 present the joint distribution for all and two GOS from GED and LED; Sections 4 displays the distribution of single GOS; section 5 demonstrates conditional distribution of GOS from GED and LED; and Section 6 summarizes the important results and offers suggestions for future research.

2 Joint Distribution of all Generalized Order Statistics

In this section, we derive the joint distribution of all GOS for generalized exponential distributions and linear exponential distributions.

2.1 Joint Distribution of all Generalized Order Statistics for Generalized Exponential Distribution

Now, we shall obtain the joint *pdf* of $X(1,n,\widetilde{m},k),\ldots,\ X(n,n,\widetilde{m},k)$ for GED and discuss some special cases.

Theorem 2.1 The joint *pdf* of $X(1,n,\widetilde{m},k),\ldots,\ X(n,n,\widetilde{m},k)$ for GED is given by

$$f(x_1,\ldots,x_n) = \prod_{i=1}^{n-1}\left(1-(1-\exp(-\lambda x_i))^{\theta}\right)^{m_i}\exp(-\lambda x_i)\,(1-\exp(-\lambda x_i))^{\theta-1}$$

$$\times\left(\prod_{j=1}^{n-1}\gamma_j\right)k\,(\theta\lambda)^n\left(1-[1-\exp(-\lambda x_n)]^{\theta}\right)^{k-1}\exp(-\lambda x_n)\,[1-\exp(-\lambda x_n)]^{\theta-1} \tag{6}$$

Proof:
Using the *pdf* and *cdf* given in (2) and (3) in (1) and collecting terms we get (6) and that completes the proof.
We discuss some special cases in Corollaries 2.2 and 2.3.

Corollary 2.2 (The joint pdf of all GOS for Exponential Distribution)
In equation(6), let $\theta=1$, the joint *pdf* of all GOS $X(1,n,\widetilde{m},k),\ldots,\ X(n,n,\widetilde{m},k)$ for Exponential distribution is

$$f(x_1,\ldots,x_n) = k\prod_{j=1}^{n-1}\gamma_j\left[\prod_{i=1}^{n-1}(1-[1-\exp(-\lambda x_i)])^{m_i}\,\lambda\exp(-\lambda x_i)\right]$$
$$\times\lambda\,(1-[1-\exp(-\lambda x_n)])^{k-1}\exp(-\lambda x_n)$$

Then

$$f(x_1,\ldots,x_n) = k\lambda\,[\exp(-\lambda x_n)]^k\prod_{j=1}^{n-1}\gamma_j\left[\prod_{i=1}^{n-1}\lambda\,[\exp(-\lambda x_i)]^{m_i+1}\right] \tag{7}$$

Corollary 2.3 (The joint pdf of all ordinary order statistics for Exponential Distribution)
In equation (7), let $k = 1$ and $m = 0$ then the joint *pdf* of all ordinary order statistics $X(1, n, 0, k), \ldots,\ X(n, n, 0, k)$ for Exponential distribution is

$$f(x_1, \ldots, x_n) = \prod_{j=1}^{n-1} \gamma_j \left[\prod_{i=1}^{n-1} \lambda \exp(-\lambda x_i) \right] \lambda \exp(-\lambda x_n)$$
$$= \prod_{j=1}^{n-1} \gamma_j \left[\prod_{i=1}^{n} \lambda \exp(-\lambda x_i) \right]$$

Given (1), let $k = 1$ and $m = 0$ then $\prod_{j=1}^{n-1} \gamma_j = \prod_{j=1}^{n-1} (n - j + 1) = n(n-1) \ldots 2.1 = n!$. Therefore, the joint *pdf* of all ordinary order statistics for Exponential Distribution is given by

$$f(x_1, \ldots, x_n) = n! \prod_{i=1}^{n} \lambda \exp(-\lambda x_i) \tag{8}$$

2.2 Joint Distribution of all Generalized Order Statistics for Linear Exponential Distribution

Now, we shall obtain the joint *pdf* of $X(1, n, \widetilde{m}, k), \ldots,\ X(n, n, \widetilde{m}, k)$ for LED and discuss some special cases.

Theorem 2.4 The joint *pdf* of $X(1, n, \widetilde{m}, k), \ldots,\ X(n, n, \widetilde{m}, k)$ for LED is given by

$$f(x_1, \ldots, x_n) = k \left(\prod_{j=1}^{n-1} \gamma_j \right) \prod_{i=1}^{n-1} \left[\left(\exp\left\{ -\left(\lambda x_i + \frac{\theta x_i^2}{2} \right) \right\} \right)^{m_i+1} (\lambda + \theta x_i) \right]$$
$$\times (\lambda + \theta x_n) \left(\exp\left\{ -\left(\lambda x_n + \frac{\theta x_n^2}{2} \right) \right\} \right)^{k} \tag{9}$$

Proof:
Using the *pdf* and *cdf* given in (4) and (5) in (1) we get

$$f(x_1, \ldots, x_n) =$$
$$\prod_{i=1}^{n-1} \left[\left(\exp\left\{ -\left(\lambda x_i + \tfrac{\theta x_i^2}{2} \right) \right\} \right)^{m_i} (\lambda + \theta x_i) \exp\left\{ -\left(\lambda x_i + \tfrac{\theta x_i^2}{2} \right) \right\} \right]$$
$$k \left(\prod_{j=1}^{n-1} \gamma_j \right) \left(\exp\left\{ -\left(\lambda x_n + \tfrac{\theta x_n^2}{2} \right) \right\} \right)^{k-1} (\lambda + \theta x_n) \exp\left\{ -\left(\lambda x_n + \tfrac{\theta x_n^2}{2} \right) \right\}$$

Collecting terms we get (9) and that completes the proof.

We discuss some special cases in Corollaries 2.5 and 2.6.

Corollary 2.5 (The joint pdf of all GOS for LED)
In equation (9), let $\theta = 1$, and collecting terms then the joint *pdf* of all GOS $X(1, n, \widetilde{m}, k), \ldots, X(n, n, \widetilde{m}, k)$ for LED is given by

$$f(x_1, \ldots, x_n) = k \left(\prod_{j=1}^{n-1} \gamma_j \right) \left[\prod_{i=1}^{n-1} \left(\exp\left\{ -\left(\lambda x_i + \tfrac{x_i^2}{2} \right) \right\} \right)^{m_i+1} (\lambda + x_i) \right] \times$$

$$\left(\exp\left\{ -\left(\lambda x_n + \frac{x_n^2}{2} \right) \right\} \right)^{k} (\lambda + x_n) \tag{10}$$

Corollary 2.6 (The joint pdf of all ordinary order statistics for LED)
In equation (10), let $k = 1$ and $m = 0$ and collecting terms then the joint pdf of all ordinary order statistics $X(1, n, 0, 1), \ldots, X(n, n, 0, 1)$ for LED is

$$f(x_1, \ldots, x_n) = n! \left[\prod_{i=1}^{n-1} \left(\exp\left\{ -\left(\lambda x_i + \tfrac{x_i^2}{2} \right) \right\} \right) (\lambda + x_i) \right] (\lambda + x_n)$$
$$\times \left(\exp\left\{ -\left(\lambda x_n + \tfrac{x_n^2}{2} \right) \right\} \right)$$

Then

$$f(x_1, \ldots, x_n) = n! \prod_{i=1}^{n} \left(\exp\left\{ -\left(\lambda x_i + \frac{x_i^2}{2} \right) \right\} \right) (\lambda + x_i) \tag{11}$$

3 Joint Distribution of Two Generalized Order Statistics

In this section, we derive the joint distribution of two GOS for generalized Exponential distribution and linear Exponential distribution.

Definition 3.1 The joint *pdf* of $i^{th}$and j^{th} GOS $X(i, n, \widetilde{m}, k)$ and $X(j, n, \widetilde{m}, k)$ and , is given by Garg (2009).

$$f_{i,j,n,\widetilde{m},k}(x_i, x_j) = \frac{c_j}{(i-1)!\,(j-i-1)!} \left[1 - F(x_i)\right]^m \left[1 - F(x_j)\right]^{\gamma_j - 1}$$

$$\times \left[g_m(F(x_i))\right]^{i-1} \left[g_m(F(x_j)) - g_m(F(x_i))\right]^{j-i-1} f(x_i)\, f(x_j), \tag{12}$$

for $0 < x_i < x_j < \infty,\ 1 \le i < j \le n$, where $c_r = \prod_{j=1}^{r} \gamma_j,\ \gamma_j = k + (n-j)(m+1)$ and

$g_m(x) = \dfrac{1 - (1-x)^{m+1}}{m+1},\ m \neq 1,\ g_m(x) = -\ln(1-x), m = -1,\ x \in (0, 1)$
Since $\lim\limits_{m \to -1} \dfrac{1 - (1-x)^{m+1}}{m+1} = -\ln(1-x)$, we write $g_m(x) = \dfrac{1 - (1-x)^{m+1}}{m+1}$ for all $x \in (0, 1)$ and for all m with $g_{-1}(x) = \lim\limits_{m \to -1} g_m(x)$.

3.1 Joint Distribution of Two Generalized Order Statistics for Generalized Exponential Distribution

Now, we shall obtain the joint pdf of two GOS for GED and discuss some special cases. In Theorem 3.2, we derive the joint pdf of $X(i,n,\widetilde{m},k)$ and $X(j,n,\widetilde{m},k)$ for GED.

Theorem 3.2 Using the *pdf* and *cdf* given in (2) and (3) in (12) we get joint *pdf* of $X(i,n,\widetilde{m},k)$ and $X(j,n,\widetilde{m},k)$ for GED is given by

$$f_{i,j,n,\widetilde{m},k}(x_i,x_j) = \frac{c_j\left[1-(1-\exp(-\lambda x_i))^{\theta}\right]^{m}\left[1-(1-\exp(-\lambda x_j))^{\theta}\right]^{\gamma_j-1}}{(i-1)!\,(j-i-1)!}$$
$$\times\left[g_m\left([1-\exp(-\lambda x_i)]^{\theta}\right)\right]^{i-1}$$
$$\times\left[g_m\left([1-\exp(-\lambda x_j)]^{\theta}\right)-g_m\left([1-\exp(-\lambda x_i)]^{\theta}\right)\right]^{j-i-1}$$
$$\times\left[(\theta\lambda)^2\exp(-\lambda(x_i+x_i))\left[(1-\exp(-\lambda x_i))(1-\exp(-\lambda x_j))\right]^{\theta-1}\right]$$

with $g_m\left([1-\exp(-\lambda x_i)]^{\theta}\right) = \dfrac{1-\left(1-[1-\exp(-\lambda x_i)]^{\theta}\right)^{m+1}}{m+1}$ and collecting terms, the joint *pdf* of $X(i,n,\widetilde{m},k)$ and $X(j,n,\widetilde{m},k)$ for GED is

$$f_{i,j,n,\widetilde{m},k}(x_i,x_j) = \frac{c_j\left[1-(1-\exp(-\lambda x_i))^{\theta}\right]^{m}\left[1-(1-\exp(-\lambda x_j))^{\theta}\right]^{\gamma_j-1}}{(i-1)!\,(j-i-1)!}$$
$$\times\left[\frac{1-\left(1-[1-\exp(-\lambda x_i)]^{\theta}\right)^{m+1}}{m+1}\right]^{i-1}$$
$$\times\left[\frac{1}{m+1}\left(\left(1-[1-\exp(-\lambda x_i)]^{\theta}\right)^{m+1}-\left(1-[1-\exp(-\lambda x_j)]^{\theta}\right)^{m+1}\right)\right]^{j-i-1}$$
$$\times\left[(\theta\lambda)^2\exp(-\lambda(x_i+x_j))\left[(1-\exp(-\lambda x_i))(1-\exp(-\lambda x_j))\right]^{\theta-1}\right] \tag{13}$$

We discuss some special cases in Corollaries 3.3 and 3.4.

Corollary 3.3 (The joint pdf of two GOS for Exponential Distribution)
In equation (13) , let $\theta=1$ and collecting terms, then the joint *pdf* of two GOS $X(i,n,\widetilde{m},k)$ and $X(j,n,\widetilde{m},k)$ for Exponential distribution is

$$f_{i,j,n,\widetilde{m},k}(x_i,x_j) = \frac{c_j\left[\exp(-\lambda x_i)\right]^{m}\left[\exp(-\lambda x_j)\right]^{\gamma_j-1}}{(i-1)!\,(j-i-1)!}$$
$$\times\lambda^2\exp\left[-\lambda(x_i+x_j)\right]\left[\frac{1-\left[\exp(-\lambda x_i)\right]^{m+1}}{m+1}\right]^{i-1}$$
$$\times\left[\frac{1}{m+1}\left((\exp(-\lambda x_i))^{m+1}-(\exp(-\lambda x_j))^{m+1}\right)\right]^{j-i-1} \tag{14}$$

Corollary 3.4 (The joint pdf of two ordinary order statistics for Exponential Distribution)
In equation (14), let $k = 1$ and $m = 0$ and collecting terms then the joint *pdf* of two ordinary order statistics $X(i, n, 0, 1)$ and $X(j, n, 0, 1)$ for Exponential distribution is

$$f_{i,j,n,0,1}(x_i, x_j) = \frac{c_j \lambda^2 \left[\exp(-\lambda x_j)\right]^{n-j} \left[1 - \exp(-\lambda x_i)\right]^{i-1} \exp\left[-\lambda(x_i + x_j)\right]}{(i-1)!\,(j-i-1)!}$$
$$\times \left[\exp(-\lambda x_i) - \exp(-\lambda x_j)\right]^{j-i-1} \tag{15}$$

where, $\gamma_j - 1 = n - j$ when $k = 1$ and $m = 0$.

3.2 Joint Distribution of Two Generalized Order Statistics for Linear Exponential Distribution

Now, we shall obtain the joint *pdf* of two GOS for LED and discuss some special cases. In Theorem 3.5, we derive the joint *pdf* of $X(i, n, \widetilde{m}, k)$ and $X(j, n, \widetilde{m}, k)$ for LED.

Theorem 3.5 Using the *pdf* and *cdf* given in (4) and (5) in (12), we get the joint *pdf* of $X(i, n, \widetilde{m}, k)$ and $X(j, n, \widetilde{m}, k)$ for LED is given by

$$f_{i,j,n,\widetilde{m},k}(x_i, x_j) = \frac{c_j \left[\exp\left(-\left[\lambda x_i + \frac{\theta x_i^2}{2}\right]\right)\right]^m \left[\exp\left(-\left[\lambda x_j + \frac{\theta x_j^2}{2}\right]\right)\right]^{\gamma_j - 1}}{(i-1)!\,(j-i-1)!}$$
$$\times \left[g_m(F(x_i))\right]^{i-1} \left[g_m(F(x_j)) - g_m(F(x_i))\right]^{j-i-1} (\lambda + \theta x_i)(\lambda + \theta x_j)$$
$$\times \exp\left(-\left[\lambda x_i + \frac{\theta x_i^2}{2}\right]\right) \exp\left(-\left[\lambda x_j + \frac{\theta x_j^2}{2}\right]\right)$$

Collecting terms, the joint *pdf* of $X(i, n, \widetilde{m}, k)$ and $X(j, n, \widetilde{m}, k)$ for LED is given by

$$f_{i,j,n,\widetilde{m},k}(x_i, x_j) = \frac{c_j \left[\exp\left(-\left[\lambda x_i + \frac{\theta x_i^2}{2}\right]\right)\right]^m \left[\exp\left(-\left[\lambda x_j + \frac{\theta x_j^2}{2}\right]\right)\right]^{\gamma_j - 1}}{(i-1)!\,(j-i-1)!}$$
$$\times \left[g_m(F(x_i))\right]^{i-1} \left[g_m(F(x_j)) - g_m(F(x_i))\right]^{j-i-1} (\lambda + \theta x_i)(\lambda + \theta x_j)$$
$$\times \exp\left(-\left[\lambda(x_i + x_j) + \frac{\theta}{2}\left(x_i^2 + x_j^2\right)\right]\right) \tag{16}$$

where,

$$g_m(F(x_i)) = \frac{1 - \left(\exp\left(-\left[\lambda x_i + \frac{\theta x_i^2}{2}\right]\right)\right)^{m+1}}{m+1} \quad \text{and } g_m(F(xj)) - g_m(F(x_i))$$
$$= \frac{\left(\exp\left(-\left[\lambda x_i + \frac{\theta x_i^2}{2}\right]\right)\right)^{m+1} - \left(\exp\left(-\left[\lambda x_j + \frac{\theta x_j^2}{2}\right]\right)\right)^{m+1}}{m+1}$$

We discuss some special cases in Corollaries 3.6 and 3.7.

Corollary 3.6 (The joint pdf of two GOS for LED)
In equation (16) , let $\theta = 1$, and collecting terms then the joint *pdf* of two GOS $X(i, n, \widetilde{m}, k)$ and $X(j, n, \widetilde{m}, k)$ for LED is given by

$$f_{i,j,n,\widetilde{m},k}(x_i, x_j) = \frac{c_j \left[\exp\left(-\left[\lambda x_i + \frac{x_i^2}{2}\right]\right)\right]^m \left[\exp\left(-\left[\lambda x_j + \frac{x_j^2}{2}\right]\right)\right]^{\gamma_j - 1}}{(i-1)!\,(j-i-1)!}$$
$$\times \left[g_m^*\left(F(x_i)\right)\right]^{i-1} \left[g_m^*\left(F(x_j)\right) - g_m^*\left(F(x_i)\right)\right]^{j-i-1} (\lambda + x_i)(\lambda + x_j)$$

$$\times \exp\left(-\left[\lambda(x_i + x_j) + \frac{1}{2}\left(x_i^2 + x_j^2\right)\right]\right) \tag{17}$$

where, $g_m^*\left(F(x_i)\right) = g_m\left(F(x_i)\right)$ when $\theta = 1$.

Corollary 3.7 (The joint pdf of two ordinary order statistics for LED)
In equation (17) , let $k = 1$ and $m = 0$, and collecting terms then the joint *pdf* of two ordinary order statistics $X(i, n, 0, 1)$ and $X(j, n, 0, 1)$ for LED is given by

$$f_{i,j,n,0,1}(x_i, x_j) = \frac{c_j (\lambda + x_i)(\lambda + x_j) \left[\left(1 - \exp\left(-\lambda x_i + \frac{x_i^2}{2}\right)\right)\right]^{i-1}}{(i-1)!\,(j-i-1)!}$$
$$\times \left[\left(\exp\left(-\left[\lambda x_j + \frac{x_j^2}{2}\right]\right) - \exp\left(-\left[\lambda x_i + \frac{x_i^2}{2}\right]\right)\right)\right]^{j-i-1}$$

$$\times \left[\exp\left(-\left[\lambda x_j + \frac{x_j^2}{2}\right]\right)\right]^{n-j} \exp\left(-\left[\lambda(x_i + x_j) + \frac{1}{2}\left(x_i^2 + x_j^2\right)\right]\right) \tag{18}$$

where $\gamma_j - 1 = n - j$ when $k = 1$ and $m = 0$.

4 Distribution of Single Generalized Order Statistics

In this section, we derive the *pdf* of the minimum and maximum GOS and consider GED and LED.

Definition 4.1 Further integrating out $x_1, x_2, \ldots, x_{r-1}, x_{r+1}, , \ldots, x_n$ from Definition 1.1, we get the *pdf* $f_{r,n,\widetilde{m},k}$ of $X(r, n, m, k)$ as given by Garg (2009),

$$f_{r,n,\widetilde{m},k}(x) = \frac{c_r}{(r-1)!} \left[1 - F(x_r)\right]^{\gamma_r - 1} g_m^{r-1}\left(F(x_r)\right) f(x_r), \tag{19}$$

where $c_r = \prod_{j=1}^{r} \gamma_j$, $\gamma_j = k + (n-j)(m+1)$ and $g_m(x) = \dfrac{1 - (1-x)^{m+1}}{m+1}$, for all $x \in (0, 1)$ and for all m with $g_{-1}(x) = \lim_{x \to -1} g_m(x)$.

Lemma 4.2 The *pdf* of the minimum generalized order statistic is

$$f_{1,n,\widetilde{m},k}(x) = [k + (n-1)(m+1)] [1 - F(x_1)]^{k+(n-1)(m+1)-1} f(x_1), \tag{20}$$

Proof:
Using (19), let $r = 1$, then $c_1 = \gamma_1 = k + (n-1)(m+1)$, we get (20) and that completes the proof.

Lemma 4.3 The *pdf* of the maximum generalized order statistic is

$$f_{n,n,\widetilde{m},k}(x) = \frac{[k + (n-1)(m+1)] [k + (n-2)(m+1)] \dots [k]}{(n-1)!}$$

$$\times [1 - F(x_n)]^{k-1} \left(\frac{1 - (1 - F(x_n))^{m+1}}{m+1} \right)^{n-1} f(x_n) \tag{21}$$

Proof:
Using Definition (4.1), let $r = n$, then $g_m(F(x_n)) = \dfrac{1 - (1 - F(x_n))^{m+1}}{m+1}$, $\gamma_n = k$, $c_n = \prod_{j=1}^{n} k + (n-j)(m+1) = [k + (n-1)(m+1)] [k + (n-2)(m+1)] \times \cdots \times [k]$,we get (21) and that completes the proof.

4.1 Distribution of Single Generalized Order Statistics for Generalized Exponential Distribution

Now, we shall obtain the *pdf* of minimum and maximum GOS for GED and discuss some special cases.

Lemma 4.4 (The pdf of the minimum GOS for GED)
Using the *pdf* and *cdf* given in (2) and (3) in (20), and collecting terms, we get the *pdf* of the minimum GOS for GED is given by

$$f_{1,n,\widetilde{m},k}(x) = [k + (n-1)(m+1)] \left[1 - (1 - \exp(-\lambda x_1))^{\theta} \right]^{k+(n-1)(m+1)-1}$$

$$\times \theta\lambda \exp(-\lambda x_1) [1 - \exp(-\lambda x_1)]^{\theta - 1} \tag{22}$$

We discuss some special cases in Corollaries 4.5 and 4.6.

Corollary 4.5 (The pdf of the minimum GOS for Exponential Distribution)
In equation (22) , let $\theta = 1$ and collecting terms, then the *pdf* of the minimum generalized order statistic for Exponential Distribution is

$$f_{1,n,\widetilde{m},k}(x) = \lambda [k + (n-1)(m+1)] [\exp(-\lambda x_1)]^{k+(n-1)(m+1)} \tag{23}$$

Corollary 4.6 (The pdf of the minimum ordinary statistics for Exponential Distribution)
In equation (23), let $k = 1$ and $m = 0$ then, $f_{1,n,0,1}(x) = n\lambda\left[\exp\left(-\lambda x_1\right)\right]^n$, *which is the well known pdf of the minimum ordinary order statistic for Exponential Distribution.*

Lemma 4.7 (The pdf of the maximum GOS for GED)
Using the *pdf* and *cdf* given in (2) and (3) in (21), and collecting terms we get *pdf* of the maximum GOS for GED,

$$f_{n,n,\widetilde{m},k}(x) = \frac{\left[k+(n-1)(m+1)\right]\left[k+(n-2)(m+1)\right]\ldots\left[k\right]}{(n-1)!}$$
$$\times\left(1-\left[1-\exp\left(-\lambda x_n\right)\right]^{\theta}\right)^{k-1}\left(\frac{1-\left(1-\left[1-\exp\left(-\lambda x_n\right)\right]^{\theta}\right)^{m+1}}{m+1}\right)^{n-1}$$
$$\times\theta\lambda\exp\left(-\lambda x_n\right)\left[1-\exp\left(-\lambda x_n\right)\right]^{\theta-1} \tag{24}$$

We discuss some special cases in Corollaries 4.8 and 4.9.

Corollary 4.8 (The pdf of the maximum GOS for Exponential Distribution)
In equation (24) , let $\theta = 1$, then the *pdf* of the maximum GOS for Exponential Distribution is given by

$$f_{n,n,\widetilde{m},k}(x) = \frac{\left[k+(n-1)(m+1)\right]\left[k+(n-2)(m+1)\right]\ldots\left[k\right]}{(n-1)!}$$
$$\times\lambda\left(\exp\left(-\lambda x_n\right)\right)^{k}\left(\frac{\left[1-\exp\left(-\lambda x_n\right)\right]^{m+1}}{m+1}\right)^{n-1} \tag{25}$$

Corollary 4.9 (The pdf of the maximum ordinary statistics for Exponential Distribution)
In equation (25), let $k = 1$ and $m = 0$, and collecting terms then the *pdf* of the maximum ordinary statistics for Exponential Distribution is given by

$$f_{n,n,0,1}(x) = n\lambda\exp\left(-\lambda x_n\right)\left[1-\exp\left(-\lambda x_n\right)\right]^{n-1} \tag{26}$$

where,
$\dfrac{\left[k+(n-1)(m+1)\right]\left[k+(n-2)(m+1)\right]\ldots\left[k\right]}{(n-1)!} = \dfrac{n(n-1)\times\cdots\times 2\times 1}{(n-1)!} = n.$
Equation (26) is the well known pdf of the maximum ordinary order statistic for Exponential Distribution.

4.2 Distribution of Single Generalized Order Statistics for Linear Exponential Distribution

Now, we shall obtain the *pdf* of minimum and maximum GOS for LED and discuss some special cases.

Lemma 4.10 (The pdf of the minimum GOS for LED)
Using the *pdf* and *cdf* given in (4) and (5) in (20), and collecting terms, the *pdf* of single GOS for LED is given by

$$f_{1,n,\widetilde{m},k}(x) = \left[k + (n-1)(m+1)\right]\left[\exp\left(-\left[\lambda x_1 + \tfrac{\theta x_1^2}{2}\right]\right)\right]^{k+(n-1)(m+1)-1}$$

$$\times (\lambda + \theta x_1) \exp\left(-\left[\lambda x_1 + \frac{\theta x_1^2}{2}\right]\right) \tag{27}$$

We discuss some special cases in Corollaries 4.11 and 4.12.

Corollary 4.11 (The pdf of the minimum GOS for LED for $\theta = 1$)
In equation (27), let $\theta = 1$ and collecting terms then the *pdf* of the minimum generalized order statistic for LED is given by

$$f_{1,n,\widetilde{m},k}(x) = \left[k + (n-1)(m+1)\right]\left[\exp\left(-\left[\lambda x_1 + \tfrac{x_1^2}{2}\right]\right)\right]^{k+(n-1)(m+1)-1}$$

$$\times (\lambda + x_1) \exp\left(-\left[\lambda x_1 + \frac{x_1^2}{2}\right]\right) \tag{28}$$

Corollary 4.12 (The pdf of the minimum ordinary statistics for LED)

In equation (28), let $k = 1$ and $m = 0$ then The *pdf* of the minimum ordinary statistics for LED is given by

$$f_{1,n,0,1}(x) = n(\lambda + x_1)\left[\exp\left(-\left[\lambda x_1 + \frac{x_1^2}{2}\right]\right)\right]^n \tag{29}$$

Equation (29) is the well known pdf of the minimum ordinary order statistic for LED.

Lemma 4.13 (The pdf of the maximum GOS for LED)
Using the *pdf* and *cdf* given in (4) and (5) in (21), and collecting terms we get the *pdf* of the maximum GOS for LED is given by

$$f_{n,n,\widetilde{m},k}(x) = \frac{\left[k + (n-1)(m+1)\right]\left[k + (n-2)(m+1)\right]\ldots[k]}{(n-1)!}$$

$$\times \left[\exp\left(-\left[\lambda x_n + \tfrac{\theta x_n^2}{2}\right]\right)\right]^{k-1}\left(\frac{1-\left(\exp\left(-\left[\lambda x_n + \tfrac{\theta x_n^2}{2}\right]\right)\right)^{m+1}}{m+1}\right)^{n-1}$$

$$\times (\lambda + \theta x_n) \exp\left(-\left[\lambda x_n + \frac{\theta x_n^2}{2}\right]\right) \quad (30)$$

We discuss some special cases in Corollaries 4.14 and 4.15.

Corollary 4.14 (The pdf of the maximum GOS for LED for)
In equation (30) , let $\theta = 1$, then the *pdf* of the maximum generalized order statistic for LED is given by

$$f_{n,n,\widetilde{m},k}(x) = \frac{[k + (n-1)(m+1)]\,[k + (n-2)(m+1)] \ldots [k]}{(n-1)!}$$

$$\times \left[\exp\left(-\left[\lambda x_n + \tfrac{x_n^2}{2}\right]\right)\right]^{k-1} \left(\frac{1 - \left(\exp\left(-\left[\lambda x_n + \tfrac{x_n^2}{2}\right]\right)\right)^{m+1}}{m+1}\right)^{n-1}$$

$$\times (\lambda + x_n) \exp\left(-\left[\lambda x_n + \frac{x_n^2}{2}\right]\right) \quad (31)$$

Corollary 4.15 (The pdf of the maximum ordinary statistics for LED)
In equation (31), let $k = 1$ and $m = 0$, and collecting terms then the *pdf* of the maximum ordinary statistics for LED is given by

$$f_{n,n,0,1}(x) = n\,(\lambda + x_n) \exp\left(-\left[\lambda x_n + \frac{x_n^2}{2}\right]\right) \left(1 - \exp\left(-\left[\lambda x_n + \frac{x_n^2}{2}\right]\right)\right)^{n-1} \quad (32)$$

where,
$\frac{[k + (n-1)(m+1)]\,[k + (n-2)(m+1)] \ldots [k]}{(n-1)!} = \frac{n\,(n-1) \times \cdots \times 2 \times 1}{(n-1)!} = n.$
Equation (32) is the well known pdf of the maximum ordinary order statistic for LED.

5 Conditional Distribution of Generalized Order Statistics

In this section, we introduce the conditional distribution of GOS and ordinary order statistic.
We consider the conditional distribution of GOS for GED and LED.

Theorem 5.1
Let $X_1,\ X_2,\ \ldots, X_n$ be a random sample from a continuous population with *cdf*, $F(x)$ and *pdf*, $f(x)$. Let $X(1, n, \widetilde{m}, k), \ldots,\ X(n, n, \widetilde{m}, k)$ denote the GOS obtained from this sample.
Then the conditional *pdf* of $X(s, n, \widetilde{m}, k)\,|X(r, n, \widetilde{m}, k) = x$ for $r < s$ is given by Samuel (2008),

$$h\left(X\left(s,n,\widetilde{m},k\right)|X\left(r,n,\widetilde{m},k\right)=x\right)=\frac{c_s\left[1-F\left(x\right)\right]^m\left[1-F\left(y\right)\right]^{\gamma_s-1}}{\left[1-F\left(x\right)\right]^{\gamma_r-1}c_r\left(s-r-1\right)!}$$

$$\times\left[g_m\left(F\left(y\right)\right)-g_m\left(F\left(x\right)\right)\right]^{s-r-1}f\left(y\right),\quad F^{-1}\left(0\right)<x\leq y<F^{-1}\left(1\right) \tag{33}$$

Note $c_r=\prod_{j=1}^{r}\gamma_j$, $\gamma_j=k+\left(n-j\right)\left(m+1\right)$ and $g_m\left(x\right)=\dfrac{1-\left(1-x\right)^{m+1}}{m+1}$ for all $x\in\left(0,1\right)$ and for all m with $g_{-1}\left(x\right)=\lim_{m\to-1}g_m\left(x\right).$

Corollary 5.2 (Conditional distribution of an ordinary order statistics)
Using Theorem (5.1), if $k=1$ and $m=0$, then

$$h\left(X\left(s,n,0,1\right)|X\left(r,n,0,1\right)\right)=\frac{c_s\left[1-F\left(y\right)\right]^{\gamma_s-1}}{\left[1-F\left(x\right)\right]^{\gamma_r-1}c_r\left(s-r-1\right)!}$$
$$\times\left[g_0\left(F\left(y\right)\right)-g_0\left(F\left(x\right)\right)\right]^{s-r-1}f\left(y\right),\quad x\left(r,n,0,1\right)\leq y\left(s,n,0,1\right)<\infty$$

$\gamma_r=n-r+1,\gamma_s=n-s+1$, $g_0\left(F\left(y\right)\right)=F\left(y\right)$, $g_0\left(F\left(x\right)\right)=F\left(x\right)$,
$c_s=n\left(n-1\right),\ldots,\left(n-s+1\right)=\dfrac{n!}{\left(n-s\right)!}$, Similarly $c_r=\dfrac{n!}{\left(n-r\right)!}$. Then

$$h\left(X\left(s,n,0,1\right)|X\left(r,n,0,1\right)\right)=\frac{\left(n-r\right)!\left[1-F\left(y\right)\right]^{n-s}\left[F\left(y\right)-F\left(x\right)\right]^{s-r-1}f\left(y\right)}{\left(n-s\right)!\left(s-r-1\right)!\left[1-F\left(x\right)\right]^{n-r}}, \tag{34}$$

which is the well known conditional probability density function of an ordinary order statistic $X\left(s,n,0,1\right)|X\left(r,n,0,1\right)$.

5.1 Conditional Distribution of GOS for Generalized Exponential Distribution

Now, we shall obtain conditional distribution of GOS for GED and discuss some special cases.

Lemma 5.3 (Conditional pdf of GOS for GED)
Using the *pdf* and *cdf* given in (2) and (3) in (33), and collecting terms, we get the the conditional *pdf* of GOS $X\left(s,n,\widetilde{m},k\right)|X\left(r,n,\widetilde{m},k\right)$ for GED is given by

$$h\left(X\left(s,n,\widetilde{m},k\right)|X\left(r,n,\widetilde{m},k\right)\right)=\frac{c_s\left[1-\left(1-\exp\left(-\lambda x\right)\right)^{\theta}\right]^m}{c_r\left(s-r-1\right)!\left[1-\left(1-\exp\left(-\lambda x\right)\right)^{\theta}\right]^{\gamma_r-1}}$$

$$\times\left[1-\left(1-\exp\left(-\lambda y\right)\right)^{\theta}\right]^{\gamma_s-1}\left[g_m\left(F\left(y\right)\right)-g_m\left(F\left(x\right)\right)\right]^{s-r-1}$$

$$\times\theta\lambda\exp\left(-\lambda y\right)\left[1-\exp\left(-\lambda y\right)\right]^{\theta-1} \tag{35}$$

where, $g_m\left(F\left(y\right)\right)-g_m\left(F\left(x\right)\right)=$

$$\frac{1}{m+1}\left[\left(1-[1-\exp(-\lambda x)]^{\theta}\right)^{m+1}-\left(1-[1-\exp(-\lambda y)]^{\theta}\right)^{m+1}\right]$$

Some special cases are derived in Corollaries 5.4 and 5.5.

Corollary 5.4 (The conditional pdf of GOS for Exponential Distribution)
In equation (35), let $\theta = 1$, and collecting terms then the conditional *pdf* of GOS $X(s,n,\widetilde{m},k)\,|X(r,n,\widetilde{m},k) = x$ for Exponential distribution is

$$h(X(s,n,\widetilde{m},k)\,|X(r,n,\widetilde{m},k)) = \frac{\lambda c_s\,[\exp(-\lambda x)]^{m-\gamma_r+1}\,[\exp(-\lambda y)]^{\gamma_s}}{c_r\,(s-r-1)!}$$

$$\times\frac{1}{m+1}\left[(\exp(-\lambda x))^{m+1}-(\exp(-\lambda y))^{m+1}\right]^{s-r-1} \tag{36}$$

Corollary 5.5 (The conditional pdf of two ordinary order statistics for Exponential Distribution)
In equation (36), let $k = 1$ and $m = 0$, and collecting terms then the conditional *pdf* of two ordinary order statistics, $X(s,n,0,1)\,|X(r,n,0,1) = x$, for Exponential distribution is

$$h(X(s,n,0,1)\,|X(r,n,0,1)) = \frac{(n-r)!\lambda\,[\exp(-\lambda x)]^{r-n}\,[\exp(-\lambda y)]^{n-s+1}}{(n-s)!\,(s-r-1)!}$$

$$\times[\exp(-\lambda x)-\exp(-\lambda y)]^{s-r-1} \tag{37}$$

where, $\dfrac{c_s}{c_r\,(s-r-1)!} = \dfrac{(n-r)!}{(n-s)!\,(s-r-1)!}$, $1-\gamma_r = r-n$, $\gamma_s = n-s+1$.
which is the well known conditional probability density function of two ordinary order statistics $X(s,n,0,1)\,|X(r,n,0,1) = x$ *for Exponential distribution.*

5.2 Conditional Distribution of GOS for Linear Exponential Distribution

Now, we shall obtain conditional distribution of GOS for LED and discuss some special cases.

Lemma 5.6 (The conditional pdf of GOS for LED)
Using the *pdf* and *cdf* given in (4) and (5) in (33), and collecting terms, the conditional *pdf* of GOS $X(s,n,\widetilde{m},k)\,|X(r,n,\widetilde{m},k) = x$ for LED is given by

$$h(X(s,n,\widetilde{m},k)\,|X(r,n,\widetilde{m},k)) =$$
$$\frac{c_s\left[(\lambda+\theta y)\exp\left(-\left[\lambda x+\frac{\theta x^2}{2}\right]\right)\right]^m\left[\exp\left(-\left[\lambda y+\frac{\theta y^2}{2}\right]\right)\right]^{\gamma_s-1}}{c_r\,(s-r-1)!\left[\exp\left(-\left[\lambda x+\frac{\theta x^2}{2}\right]\right)\right]^{\gamma_r-1}}$$

$$\times \left[g_m\left(F\left(y\right)\right) - g_m\left(F\left(x\right)\right)\right]^{s-r-1} \exp\left(-\left[\lambda y + \frac{\theta y^2}{2}\right]\right) \tag{38}$$

where, $g_m\left(F\left(y\right)\right) - g_m F\left(x\right) =$
$\dfrac{1}{m+1}\left[\left(\exp\left(-\left[\lambda x + \frac{\theta x^2}{2}\right]\right)\right)^{m+1} - \left(\exp\left(-\left[\lambda y + \frac{\theta y^2}{2}\right]\right)\right)^{m+1}\right]$

Some special cases are derived in Corollaries 5.7 and 5.8.

Corollary 5.7 (The conditional pdf of GOS for LED)
In equation (38), let $\theta = 1$, and collecting terms then the conditional *pdf* of GOS $X\left(s,n,\widetilde{m},k\right)|X\left(r,n,\widetilde{m},k\right) = x$ for LED is

$$h\left(X\left(s,n,\widetilde{m},k\right)|X\left(r,n,\widetilde{m},k\right)\right) = \frac{\left[\exp\left(-\left[\lambda x + \frac{x^2}{2}\right]\right)\right]^m \exp\left(-\left[\lambda y + \frac{y^2}{2}\right]\right)}{c_r\left(s-r-1\right)!\left[\exp\left(-\left[\lambda x + \frac{x^2}{2}\right]\right)\right]^{\gamma_r - 1}}$$

$$\times c_s\left(\lambda + y\right)\left[g_m^*\left(F\left(y\right)\right) - g_m^*\left(F\left(x\right)\right)\right]^{s-r-1}\left[\exp\left(-\left[\lambda y + \frac{y^2}{2}\right]\right)\right]^{\gamma_s - 1} \tag{39}$$

where, $g_m^*\left(F\left(y\right)\right) - g_m^* F\left(x\right) =$
$\dfrac{1}{m+1}\left[\left(\exp\left(-\left[\lambda x + \frac{x^2}{2}\right]\right)\right)^{m+1} - \left(\exp\left(-\left[\lambda y + \frac{y^2}{2}\right]\right)\right)^{m+1}\right]$

Corollary 5.8 (The conditional pdf of two ordinary order statistics for LED)
In equation (39), let $k = 1$ and $m = 0$, and collecting terms then the conditional *pdf* of two ordinary order statistics, $X\left(s,n,0,1\right)|X\left(r,n,0,1\right) = x$, for LED is

$$h\left(X\left(s,n,0,1\right)|X\left(r,n,0,1\right)\right) = \frac{\left(\lambda + y\right)\left(n-r\right)!\left[\exp\left(-\left[\lambda y + \frac{y^2}{2}\right]\right)\right]^{n-s+1}}{\left(n-s\right)!\left(s-r-1\right)!\left[\exp\left(-\left[\lambda x + \frac{x^2}{2}\right]\right)\right]^{n-r}}$$

$$\times\left[\exp\left(-\left[\lambda x + \frac{x^2}{2}\right]\right) - \exp\left(-\left[\lambda y + \frac{y^2}{2}\right]\right)\right]^{s-r-1} \tag{40}$$

which is the well known conditional probability density function of two ordinary order statistics $X\left(s,n,0,1\right)|X\left(r,n,0,1\right) = x$ *for Linear Exponential distribution.*

6 Conclusion and Future Research

In this paper, we have derived the joint *pdfs* of GOS for Generalized and Linear Exponential distributions in explicit forms. In addition, the *pdf* of the conditional distribution of GOS from those distributions is given. Furthermore, some special cases have been discussed.

Many opportunities of future research are available. The plan for the future research on GOS from Generalized and Linear Exponential distributions can be split into two main areas. Estimation and hypothesis testing of Generalized Exponential parameters based on generalized order statistics.

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